Retake Test 1 Febr. 2022

a i Def abs cond. number $\max\limits_{\delta d} \frac{\|\delta x\|}{\|\delta d\|}$

where $F(x+\delta x) = d + \delta d \rightarrow A(x+\delta x) = d + \delta d$

$F(x) = d \qquad\qquad A x = d$

$\frac{\partial F}{\partial x} \qquad\qquad A\,\delta x = \delta d$

$$\Rightarrow \max_{\delta d} \frac{\|\delta x\|}{\|\delta d\|} = \max_{\delta d} \frac{\|A^{-1}\delta d\|}{\|\delta d\|} = \|A^{-1}\|$$

ii Def rel. cond. number $\max\limits_{\delta d} \frac{\|\delta x\| / \|x\|}{\|\delta d\| / \|d\|} = \frac{\|d\|}{\|x\|} \max\limits_{\delta d} \frac{\|\delta x\|}{\|\delta d\|}$

$$= \frac{\|d\|}{\|x\|} \|A^{-1}\|$$

iii $\kappa(A) = \|A\| \|A^{-1}\|$

$$\max_{d} \underbrace{\frac{\|d\|}{\|x\|}} \|A^{-1}\| = \max_{x} \frac{\|d\|}{\|x\|} \|A^{-1}\| = \max_{x} \frac{\|Ax\|}{\|x\|} \|A^{-1}\|$$

x and d are 1-1 coupled through $Ax = d$

$$= \|A\| \|A^{-1}\| = \kappa(A)$$

So the condition number $\kappa(A)$ is the relative condition for the d that gives the biggest propagation of the error.

b i) $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $A_1 = \begin{pmatrix} 2.001 & 2 \\ 2 & 2.001 \end{pmatrix}$ $A_2 = \begin{pmatrix} 2.001 & -2 \\ -2 & 2.001 \end{pmatrix}$

- $\det(A_1 - \lambda I) = \det \begin{pmatrix} 2.001-\lambda & 2 \\ 2 & 2.001-\lambda \end{pmatrix} = (2.001-\lambda)^2 - 4$

$\det(A_2 - \lambda I) = \det \begin{pmatrix} 2.001-\lambda & -2 \\ -2 & 2.001-\lambda \end{pmatrix} = (2.001-\lambda)^2 - 4$

the characteristic polynomials of A_1 and A_2 are similar

$\Rightarrow$ eigenvalues of A_1 and A_2 are similar

$\Rightarrow (2.001-\lambda)^2 - 4 = 0 \Rightarrow \lambda_{\pm} = \pm 2 + 2.001$

- eigenvectors

A_1 : • $\lambda = 4.001 \Rightarrow (A_1 - \lambda I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -2x_1 + 2x_2 = 0 \Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

• $\lambda = 0.001 \Rightarrow \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

A_2 : • $\lambda = 4.001 \Rightarrow \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 0.001 \Rightarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b

i Same characteristic polynomial
+ checking that the eigenv. are the same } see previous page

more advanced $A_1 = 2.001\, I + \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

$A_2 = \sim - \sim$

Any polynomial of $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ has the same eigenvectors so A_1 and A_2 have

Eigenvalues are 0.001 an 4.001

ii both A_1 & A_2 are symmetric $\to$ $\|A_1\|_2 = \rho(A_1)$;
$\|A_2\|_2 = \rho(A_2)$

$\|A_1^{-1}\| = \rho(A_1^{-1}) = \max\left(\left|\frac{1}{\lambda_1}\right|, \left|\frac{1}{\lambda_2}\right|\right) =$

$\|A_2^{-1}\| = \rho(A_2^{-1}) = \max$ — —

$$\Rightarrow \quad \kappa_2(A_1) = \kappa_2(A_2)$$

iii Relative condition number is here

$$\frac{\|d\|}{\|X\|}\|A^{-1}\| =$$

$d = b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector

$$A_1 d = 4.001\, d \rightarrow A_1 \underbrace{\frac{d}{4.001}}_{X} = d \rightarrow \frac{\|d\|}{\|X\|} = \frac{\|d\|}{\|\frac{d}{4.001}\|}$$

$$A_2 d = 0.001\, d \rightarrow A_2 \underbrace{\frac{d}{0.001}}_{X} = d \rightarrow \frac{\|d\|}{\|X\|} = \frac{\|d\|}{\|\frac{d}{0.001}\|}$$

So for A_1 the relative cond number is

$$\underbrace{4.001}_{\|d\|/\|X\|} \underbrace{\|A_1^{-1}\|}_{} = 4.001 \times 1000 = 4001$$

$$\hookrightarrow \|A_1^{-1}\| = \max\left(\frac{1}{|\lambda_1|}, \frac{1}{|\lambda_2|}\right) = \max\left(\frac{1}{0.001}, \frac{1}{4.001}\right)$$

For A_2 the relative cond number is

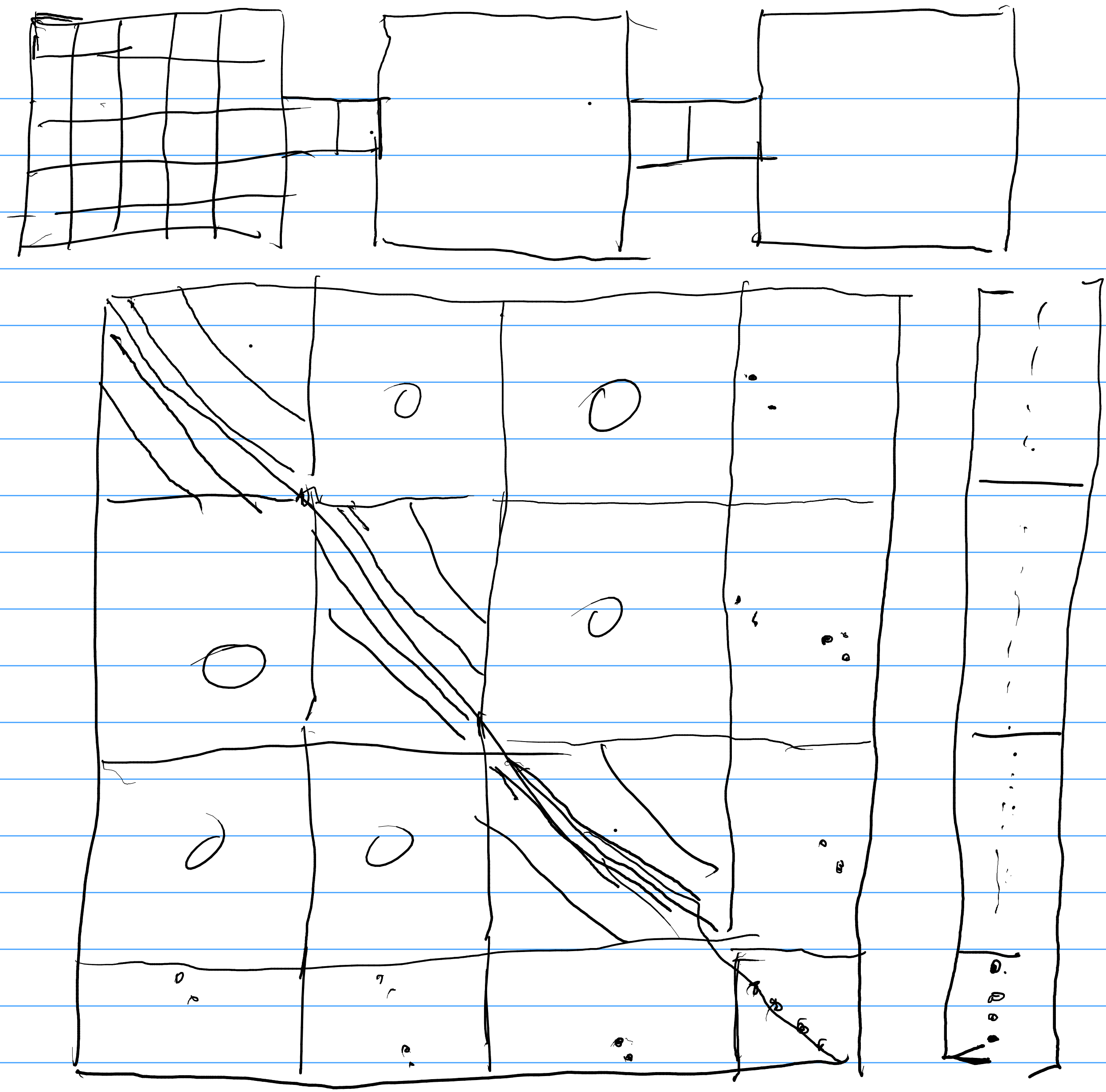
$0.001 \, \| A_2^{-1} \| = 0.001 \times 1000 = 1$

So for this problem the solution of

$A_1 x = d$ will suffer most from

round-off error propagation

Problem 2



The advantage of this ordering is that we reduce the fill in in the L and U

We need three LU factorizations on the blocks which can be made independently of each other. After the elimination of the associated unknowns there remains only a 4x4 matrix. Having solved for the associated unknowns we can solve for the unknowns in the blocks independently.